Journal of Approximation Theory **98**, 72–103 (1999)
Article ID jath.1998.3279, available online at http://www.idealibrary.com on IDEAL®

Multidimensional Periodic Multiwavelets

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Communicated by Rong-Qing Jia

Received May 27, 1997; accepted in revised form April 30, 1998

Necessary and sufficient conditions are given for the convergence of infinite products of matrices of complex numbers. The results are applied to the solution of periodic matrix refinement equations. Conditions are given for the solutions to be in $L^2([0, 2\pi)^s)$ and generate a multiresolution of multiplicity r. A general algorithm for constructing multidimensional periodic multiwavelets from a scaling vector which generates a multiresolution is also given.

Key Words: periodic multiresolution with multiplicity r and dilation matrix M; infinite matrix product; matrix extension; multidimensional periodic multiwavelets.

1. INTRODUCTION

Let s be a positive integer and M be an $s \times s$ matrix with integer entries such that all its eigenvalues lie outside the unit circle. For $k \geqslant 0$, let $\mathscr{L}_k$ denote a full collection of coset representatives of $\mathbb{Z}^s/M^k\mathbb{Z}^s$ and $\mathscr{R}_k$ denote a full collection of coset representatives of $\mathbb{Z}^s/D^k\mathbb{Z}^s$ where

$$D = M^T. \tag{1.1}$$

Then

$$\mathbb{Z}^s = \bigcup_{\ell \in \mathscr{L}_k} (\ell + M^k\mathbb{Z}^s) = \bigcup_{j \in \mathscr{R}_k} (j + D^k\mathbb{Z}^s),$$

and for any distinct $\ell_1, \ell_2 \in \mathscr{L}_k$, $j_1, j_2 \in \mathscr{R}_k$,

$$(\ell_1 + M^k\mathbb{Z}^s) \cap (\ell_2 + M^k\mathbb{Z}^s) = \varnothing = (j_1 + D^k\mathbb{Z}^s) \cap (j_2 + D^k\mathbb{Z}^s).$$

As an illustration, for $M = D = 2I_s$ where I_s is the $s \times s$ identity matrix, we may take $\mathscr{L}_k = \mathscr{R}_k = \{0, 1, ..., 2^k - 1\}^s$.

0021-9045/99 $30.00

Throughout the paper, we shall denote

$$d := |\det(M)| = |\det(D)|. \tag{1.2}$$

Then $d^k = |\mathscr{L}_k| = |\mathscr{R}_k|$.

A sequence of subspaces $\{V_k : k \geqslant 0\}$ of $L^2([0, 2\pi)^s)$ is a *periodic multiresolution* (MR) *of* $L^2([0, 2\pi)^s)$ *with multiplicity* r *and dilation matrix* M if it satisfies the following conditions:

MR1. For $k = 0, 1, ...,$ $\dim V_k = r\,|\det(M^k)|$ and there exist functions $\phi_k^m \in V_k$, $m = 1, 2, ..., r$, such that $\{T_k^\ell \phi_k^m : m = 1, 2, ..., r,\ \ell \in \mathscr{L}_k\}$ is a basis for V_k, where $T_k^\ell f := f(\cdot - 2\pi M^{-k}\ell)$.

MR2. For $k = 0, 1, ...,$ $V_k \subseteq V_{k+1}$.

MR3. $\overline{\bigcup_{k \geqslant 0} V_k} = L^2([0, 2\pi)^s)$.

The functions ϕ_k^m, $k \geqslant 0$, $m = 1, 2, ..., r$, are called *scaling functions*, and $\phi_k := (\phi_k^1, ..., \phi_k^r)^T$, $k \geqslant 0$, are called *scaling vectors*. The scaling vectors ϕ_k are said to generate the multiresolution $\{V_k : k \geqslant 0\}$. A general theory of periodic multiresolutions and wavelets for the one-dimensional scalar case ($s = r = 1$) with dilation $M = 2$ was given in [18]. Periodic spline wavelets and trigonometric wavelets were earlier studied in [8, 24, and 26]. More results on the theory and its applications can be found in recent literatures (see [2–6, 23, 28]). This paper develops the corresponding theory for multiresolutions of multiplicity r with dilation matrix M and the construction of the corresponding multidimensional multiwavelets.

If the sequence ϕ_k generates a multiresolution of multiplicity r with dilation matrix M, then $k = 0, 1, ...,$

$$\phi_k = \sum_{\ell \in \mathscr{L}_{k+1}} H_{k+1}(\ell)\, T_{k+1}^\ell \phi_{k+1}, \tag{1.3}$$

where $H_{k+1} \in \mathscr{S}(M^{k+1})^{r \times r}$, the class of periodic sequences of $r \times r$ complex matrices of period M^{k+1}, that is, $H_{k+1}(\ell + M^{k+1}p) = H_{k+1}(\ell)$ for all $\ell, p \in \mathbb{Z}^s$.

For $H_{k+1} \in \mathscr{S}(M^{k+1})^{r \times r}$, we define the finite Fourier transform of H_{k+1} by

$$\hat{H}_{k+1}(j) = \sum_{\ell \in \mathscr{L}_{k+1}} H_{k+1}(\ell)\, e^{-ij \cdot (2\pi M^{-(k+1)}\ell)}, \qquad j \in \mathscr{R}_{k+1}, \tag{1.4}$$

where the dot in the exponent of e denotes scalar product. Since $D = M^T$, it follows that $\hat{H}_{k+1} \in \mathscr{S}(D^{k+1})^{r \times r}$. On the other hand, if (1.4) holds, then

$$H_{k+1}(\ell) = \frac{1}{|\mathscr{R}_{k+1}|} \sum_{j \in \mathscr{R}_{k+1}} \hat{H}_{k+1}(j)\, e^{i\ell \cdot (2\pi D^{-(k+1)}j)}, \qquad \ell \in \mathscr{L}_{k+1}. \tag{1.5}$$

Thus the finite Fourier transform (1.4) and its inverse (1.5) provide a one-to-one correspondence between $\mathcal{S}(M^{k+1})^{r\times r}$ and $\mathcal{S}(D^{k+1})^{r\times r}$. The formula (1.5) is obtained by using the relation (see [12])

$$\sum_{j\in\mathcal{R}_{k+1}} e^{i(\ell-\nu)\cdot(2\pi D^{-(k+1)}j)} = \begin{cases} |\mathcal{R}_{k+1}|, & \text{if } \ell=\nu, \\ 0, & \text{if } \ell\neq\nu, \end{cases} \tag{1.6}$$

for $\ell, \nu\in\mathcal{L}_{k+1}$. Note that the relation (1.6) also yields

$$\frac{1}{|\mathcal{R}_{k+1}|}\sum_{j\in\mathcal{R}_{k+1}} |\hat{\alpha}(j)|^2 = \sum_{\ell\in\mathcal{L}_{k+1}} |\alpha(\ell)|^2, \qquad \alpha\in\mathcal{S}(M^{k+1}), \tag{1.7}$$

where $\mathcal{S}(M^{k+1}) := \mathcal{S}(M^{k+1})^{1\times 1}$.

By taking Fourier coefficients, we see that (1.3) is equivalent to

$$\hat{\phi}_k(n) = \hat{H}_{k+1}(n)\,\hat{\phi}_{k+1}(n), \qquad n\in\mathbb{Z}^s, \tag{1.8}$$

for $k\geqslant 0$. Thus the first step in the construction of a periodic multiresolution with multiplicity r and dilation matrix M is to find a sequence of vectors $a_k\in\ell^2(\mathbb{Z}^s)^r$ satisfying

$$a_k(n) = \hat{H}_{k+1}(n)\,a_{k+1}(n), \qquad n\in\mathbb{Z}^s, \tag{1.9}$$

for a given sequence of matrices $H_{k+1}\in\mathcal{S}(M^{k+1})^{r\times r}$. If such a sequence of vectors a_k exists, then for $k\geqslant 0$, a_k is the Fourier sequence of a scaling vector $\phi_k\in L^2([0, 2\pi)^s)^r$ that satisfies (1.3). Equation (1.3) or (1.9) is a periodic analogue of the matrix refinement equation

$$\Phi(x) = \sum_{\ell\in\mathbb{Z}^s} H(\ell)\,\Phi(Mx-\ell), \qquad x\in\mathbb{R}^s, \tag{1.10}$$

which is the subject of intensive study in wavelet analysis and subdivision processes. We shall call (1.3) or (1.9) a *periodic matrix refinement equation.*

Solutions of the matrix refinement equation (1.10) with a finitely supported mask $H(\ell)$ and their properties are the main focus of present research in multiwavelets (see [7, 9, 14, 15, 17, 20, 22, 27, 29]). In this case, many properties of the solutions can be characterized in terms of the spectrum of the corresponding transition operators. Finitely supported masks give compactly supported scaling vectors. A general algorithm for the construction of compactly supported univariate multiwavelets from a given compactly supported scaling vector for the nonperiodic case can be found in [19]. We remark that such a result on scaling vectors and multiwavelet construction is not available for infinitely supported masks and non-compactly supported multiwavelets. Our object here is the construction of periodic scaling vectors and the corresponding multiwavelets. Our results cover both the periodic analogue of infinitely supported multivariate scaling vectors and multiwavelets as well as the finitely supported ones.

The solution of (1.9) depends on the analysis of the convergence of an infinite product of matrices. This is done in Section 2, where necessary and sufficient conditions for the convergence of infinite products of matrices of complex numbers are first given and then applied to the solution of (1.9). We also give conditions on $\hat{H}_{k+1}$ in order that the solution of (1.9) leads to scaling vectors in $L^2([0, 2\pi)^s)^r$. The rest of the paper is organized as follows. Section 3 deals with the linear independence of the shifts of the scaling functions, and presents a necessary and sufficient condition for MR3 to be satisfied. The corresponding multiwavelets are constructed in Section 4. Our analysis is enriched by a class of linearly independent functions which we call *polyphase splines*, which is an extension of the *orthogonal splines* introduced in [18]. We conclude the paper with a construction of a family of periodic box spline wavelets in a three-direction mesh.

2. SOLUTIONS OF PERIODIC REFINEMENT EQUATIONS

We shall first establish a result on the uniform convergence of infinite product of matrices of complex numbers. A matrix is said to satisfy *Condition E^** if 1 is a nondegenerate eigenvalue, and all its other eigenvalues lie inside the unit circle. Equivalently, A satisfies Condition E^* if and only if A^n converges to a nonzero matrix as $n \to \infty$.

THEOREM 2.1. *Let A_ℓ, $\ell \geqslant 1$, be a sequence of $r \times r$ matrices of complex numbers, and suppose that there exists an $r \times r$ matrix A such that*

$$\sum_{\ell=1}^{\infty} \|A_\ell - A\| < \infty. \tag{2.1}$$

Then the matrix product

$$P_{k,n} := \prod_{\ell=k+1}^{k+n} A_\ell \tag{2.2}$$

converges uniformly in k as $n \to \infty$, and

$$\lim_{k \to \infty} \lim_{n \to \infty} P_{k,n} \text{ exists and is nonzero,} \tag{2.3}$$

if and only if A satisfies Condition E^*.

Proof. Since the matrix A satisfies Condition E^*, its spectral radius $\rho(A)=1$, and the number 1 is its only eigenvalue on the unit circle and is nondegenerate. Therefore, there is a matrix norm $\|\cdot\|$ such that $\|A\|=1$. For all $k \geqslant 0$, $n \geqslant 1$,

$$\left\| \prod_{\ell=k+1}^{k+n} A_\ell \right\| \leqslant \prod_{\ell=k+1}^{k+n} (\|A_\ell - A\| + 1) \leqslant \prod_{\ell=k+1}^{k+n} \exp(\|A_\ell - A\|)$$
$$= \exp\left(\sum_{\ell=k+1}^{k+n} \|A_\ell - A\| \right) \leqslant C < \infty,$$

where C is an absolute constant $\geqslant 1$. It follows that for any $k \geqslant 0$, $n \geqslant 1$, and for any nonnegative $m < n$,

$$\left\| \prod_{\ell=k+1}^{k+n} A_\ell - \left(\prod_{\ell=k+1}^{k+m} A_\ell \right) A^{n-m} \right\| \leqslant C^2 \sum_{\ell=k+m+1}^{k+n} \|A_\ell - A\|. \tag{2.4}$$

Since A satisfies Condition E^*, the unitary space $\mathbb{C}^r$ can be written as $\mathbb{C}^r = E_1 \oplus Q$, where E_1 is the eigenspace of A corresponding to the eigenvalue 1, and the spectral radius $\rho(A|_Q) < 1$. Take any $x \in \mathbb{C}^r$ and write $x = v + w$, where $v \in E_1$ and $w \in Q$. The inequality (2.4) gives

$$\left\| \left(\prod_{\ell=k+1}^{k+n} A_\ell \right) v - \left(\prod_{\ell=k+1}^{k+m} A_\ell \right) v \right\| = \left\| \left(\prod_{\ell=k+1}^{k+n} A_\ell \right) v - \left(\prod_{\ell=k+1}^{k+m} A_\ell \right) A^{n-m} v \right\|$$
$$\leqslant C^2 \|v\| \sum_{\ell=k+m+1}^{k+n} \|A_\ell - A\|,$$

which tends to zero uniformly in k as $m, n \to \infty$. Hence, $(\prod_{\ell=k+1}^{k+n} A_\ell)\, v$ converges uniformly in k as $n \to \infty$. Similarly, (2.4) gives

$$\left\| \left(\prod_{\ell=k+1}^{k+n} A_\ell \right) w - \left(\prod_{\ell=k+1}^{k+m} A_\ell \right) A^{n-m} w \right\| \leqslant C^2 \|w\| \sum_{\ell=k+m+1}^{k+n} \|A_\ell - A\|.$$

Since $A^{n-m} w \to 0$ as $n - m \to \infty$, this means that

$$\left(\prod_{\ell=k+1}^{k+n} A_\ell \right) w \to 0, \qquad \text{as} \quad n \to \infty.$$

Thus, for any $x = v + w \in \mathbb{C}^r$, $(\prod_{\ell=k+1}^{k+n} A_\ell)\, x$ converges uniformly in k as $n \to \infty$.

Now, with $m = 0$, (2.4) becomes

$$\left\| \prod_{\ell=k+1}^{k+n} A_\ell - A^n \right\| \leqslant C \sum_{\ell=k+1}^{k+n} \|A_\ell - A\|.$$

Taking limits as $n \to \infty$ gives

$$\left\| \prod_{\ell=k+1}^{\infty} A_\ell - P \right\| \leqslant C \sum_{\ell=k+1}^{\infty} \|A_\ell - A\|,$$

where $P := \lim_{n\to\infty} A^n$. It follows that

$$\lim_{k\to\infty} \prod_{\ell=k+1}^{\infty} A_\ell = P,$$

and P is nonzero since A satisfies Condition E^*.

Conversely, suppose that the matrix product $P_{k,n} := \prod_{\ell=k+1}^{k+n} A_\ell$ converges uniformly in k as $n \to \infty$, and

$$\lim_{k\to\infty} \lim_{n\to\infty} P_{k,n} = P,$$

where P is a nonzero matrix. Then

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} \lim_{k\to\infty} \prod_{\ell=k+1}^{k+n} A_\ell = P. \tag{2.5}$$

Hence A satisfies Condition E^*. ∎

COROLLARY 2.1. *For $\ell \geqslant 1$, let $\hat{H}_\ell \in \mathscr{S}(D^\ell)^{r\times r}$, and suppose that for each $n \in \mathbb{Z}^s$, there exists an $r \times r$ matrix $\hat{H}(n)$ such that*

$$\sum_{\ell=1}^{\infty} \|\hat{H}_\ell(n) - \hat{H}(n)\| \tag{2.6}$$

converges. Then the infinite matrix product $\prod_{\ell=k+1}^{\infty} \hat{H}_\ell(n)$ converges uniformly for $k \geqslant 0$, and

$$\lim_{k\to\infty} \prod_{\ell=k+1}^{\infty} \hat{H}_\ell(n) \text{ exists and is nonzero},$$

if and only if the matrix $\hat{H}(n)$ satisfies Condition E^*.

Furthermore, if $\hat{H}(n)$ satisfies Condition E^*, then

$$\lim_{k\to\infty} \prod_{\ell=k+1}^{\infty} \hat{H}_\ell(n) = \lim_{m\to\infty} \hat{H}(n)^m. \tag{2.7}$$

Theorem 2.1 has other applications. For instance, by taking $A_\ell(u) := A(u/2^\ell)$, $\ell \geqslant 1$ and $u \in \mathbb{R}^s$, where $A(u)$ is an $r \times r$ matrix of functions, we obtain the following general result on the convergence of infinite matrix product for the solution of matrix refinement equations (see [14]) under very weak conditions.

COROLLARY 2.2. *Let* $A(u)$, $u \in \mathbb{R}^s$, *be an* $r \times r$ *matrix of continuous functions such that*

$$\sum_{\ell=1}^{\infty} \|A(u/2^{\ell}) - A(0)\| < \infty, \qquad u \in \mathbb{R}^s, \tag{2.8}$$

where the convergence is uniform on compact sets. Then the infinite matrix product $\prod_{\ell=k+1}^{\infty} A(u/2^{\ell})$ *converges locally uniformly in u and uniformly in k, and*

$$\lim_{k \to \infty} \prod_{\ell=k+1}^{\infty} A(u/2^{\ell}) \text{ exists and is nonzero,}$$

if and only if $A(0)$ *satisfies Condition* E^*.

Furthermore, if $A(0)$ *satisfies Condition* E^*, *then*

$$\lim_{k \to \infty} \prod_{\ell=k+1}^{\infty} A(u/2^{\ell}) = \lim_{m \to \infty} A(0)^m. \tag{2.9}$$

By Theorem 2.1, if A satisfies Condition E^*, then for any 1-eigenvector of A,

$$\left(\prod_{\ell=k+1}^{k+n} A_{\ell}\right) v \to \left(\prod_{\ell=k+1}^{\infty} A_{\ell}\right) v$$

uniformly in k as $n \to \infty$. We now extend this result to a matrix A with spectral radius $\rho(A) \geqslant 1$. If A has an eigenvalue λ with $|\lambda| = \rho(A)$ such that A/λ satisfies Condition E^*, then we say that A satisfies *Condition* $E(\lambda)^*$.

THEOREM 2.2. *Let* A_{ℓ}, $\ell \geqslant 1$, *be a sequence of* $r \times r$ *matrices of complex numbers, and suppose that there exists an* $r \times r$ *matrix* A *such that*

$$\sum_{\ell=1}^{\infty} \rho(A)^{\ell} \|A_{\ell} - A\| < \infty. \tag{2.10}$$

If A *satisfies Condition* $E(\lambda)^*$ *and* v *is an* 1-*eigenvector of* A, *then*

$$\left(\prod_{\ell=k+1}^{k+n} A_{\ell}\right) v \to \left(\prod_{\ell=k+1}^{\infty} A_{\ell}\right) v \tag{2.11}$$

uniformly in k as $n \to \infty$, and

$$\lim_{k \to \infty} \left(\prod_{\ell = k+1}^{\infty} A_\ell \right) v = v. \tag{2.12}$$

Proof. For all integers $k \geqslant 0$, $n \geqslant 1$ and any 1-eigenvector v of A,

$$\begin{aligned}\left\| \left(\prod_{\ell=k+1}^{k+n} A_\ell \right) v - \left(\prod_{\ell=k+1}^{k+n-1} A_\ell \right) v \right\| &= \left\| \left(\prod_{\ell=k+1}^{k+n} A_\ell \right) v - \left(\prod_{\ell=k+1}^{k+n-1} A_\ell \right) Av \right\| \\ &\leqslant \left\| \prod_{\ell=k+1}^{k+n-1} A_\ell \right\| \|v\| \, \|A_{k+n} - A\| \\ &\leqslant C\,|\lambda|^{n-1} \, \|A_{k+n} - A\|,\end{aligned}$$

where C is an absolute constant. Thus for $m < n$,

$$\left\| \left(\prod_{\ell=k+1}^{k+n} A_\ell \right) v - \left(\prod_{\ell=k+1}^{k+m} A_\ell \right) v \right\| \leqslant C\,|\lambda|^{-k-1} \sum_{\ell=m+1}^{n} |\lambda|^{k+\ell} \, \|A_{k+\ell} - A\|, \tag{2.13}$$

and so (2.11) holds by (2.10).

By (2.13) with $m = 0$,

$$\begin{aligned}\left\| \left(\prod_{\ell=k+1}^{k+n} A_\ell \right) v - v \right\| &= \left\| \left(\prod_{\ell=k+1}^{k+n} A_\ell \right) v - A^n v \right\| \\ &\leqslant C\,|\lambda|^{-k-1} \sum_{\ell=k+1}^{k+n} |\lambda|^{\ell} \, \|A_\ell - A\|.\end{aligned}$$

Letting $n \to \infty$ and then taking the limit as $k \to \infty$ gives

$$\lim_{k \to \infty} \left(\prod_{\ell=k+1}^{\infty} A_\ell \right) v = \lim_{n \to \infty} A^n v = v,$$

and the proof is complete. ▮

COROLLARY 2.3. *Let $A(u)$, $u \in \mathbb{R}^s$, be an $r \times r$ matrix of continuous functions such that $A(0)$ satisfies Condition $E(\lambda)^*$ where $|\lambda| = \rho(A(0)) \geqslant 1$, and*

$$\sum_{\ell=1}^{\infty} |\lambda|^{\ell} \, \|A(u/(|\lambda| + \varepsilon)^{\ell}) - A(0)\| < \infty, \qquad u \in \mathbb{R}^s, \tag{2.14}$$

for some $\varepsilon > 0$, *where the convergence is uniform on compact sets. If* v *is an* 1-*eigenvector of* $A(0)$, *then* $(\prod_{\ell=k+1}^{\infty} A(u/(|\lambda|+\varepsilon)^{\ell}))\, v$ *converges locally uniformly in* u *and uniformly in* k, *and*

$$\lim_{k\to\infty}\left(\prod_{\ell=k+1}^{\infty} A(u/(|\lambda|+\varepsilon)^{\ell})\right) v = v. \tag{2.15}$$

Remark 1. If $|\lambda| = \rho(A(0)) < 2$, the results of Corollary 2.3 hold for $|\lambda| + \varepsilon = 2$. This special case was established in [14] and [9] under a stronger assumption than (2.14).

We now consider conditions on H_{k+1} that guarantee a solution ϕ_k of (1.3) in $L^2([0, 2\pi)^s)^r$. To this end, we assume that for each $n \in \mathbb{Z}^s$, there exists an $r\times r$ matrix $\hat{H}(n)$ that satisfies Condition E^*, and that the hypothesis of Corollary 2.1 is satisfied. Then the infinite matrix product $\prod_{\ell=k+1}^{\infty} \hat{H}_{\ell}(n)$ exists by Corollary 2.1. Let $v(n) \in \mathbb{C}^r$ be a sequence of non-zero vectors such that

$$\|v(n)\| \leqslant K \qquad \text{for all} \quad n \in \mathbb{Z}^s, \tag{2.16}$$

for some constant $K > 0$. Define a sequence of vectors $a_k(n) \in \mathbb{C}^r$ by

$$a_k(n) := \left(\prod_{\ell=k+1}^{\infty} \hat{H}_{\ell}(n)\right) v(n). \tag{2.17}$$

Then it follows that $a_k(n) = \hat{H}_{k+1}(n)\, a_{k+1}(n)$. Since

$$\lim_{k\to\infty} \prod_{\ell=k+1}^{\infty} \hat{H}_{\ell}(n) = \lim_{m\to\infty} \hat{H}(n)^m =: P(n),$$

it also follows that

$$\lim_{k\to\infty} a_k(n) = P(n)\, v(n).$$

Furthermore, $P(n)\, v(n)$ is an eigenvector of $\hat{H}(n)$ with eigenvalue 1. If 1 is a simple eigenvalue, then $\hat{\phi}_k = a_k$ is the unique solution (up to a constant multiple) of (1.8).

The following theorem gives a sufficient condition for a_k to be in $\ell^2(\mathbb{Z}^s)^r$.

THEOREM 2.3. *Suppose that for* $k \geqslant 0$,

$$\sum_{\ell\in\mathscr{R}_1} \|\hat{H}_{k+1}(j + D^k \ell)\|^2 \leqslant 1, \qquad j \in \mathscr{R}_k, \tag{2.18}$$

for some operator matrix norm. Then

$$\sum_{p \in \mathbb{Z}^s} \|a_k(j + D^k p)\|^2 \leqslant K^2, \tag{2.19}$$

for all $j \in \mathscr{R}_k$.

Proof. First, observe that (2.18) implies that for $k \geqslant 0$,

$$\sum_{\ell \in \mathscr{R}_1} \|\hat{H}_{k+1}(n + D^k \ell)\|^2 \leqslant 1, \qquad n \in \mathbb{Z}^s, \tag{2.20}$$

where $\mathscr{R}_1$ is any full collection of coset representatives of $\mathbb{Z}^s / D\mathbb{Z}^s$.

We shall show that for any $n \in \mathbb{Z}^s$ and any full collection $\mathscr{R}_k$ of coset representatives of $\mathbb{Z}^s / D^k \mathbb{Z}^s$, if $m \geqslant k+1$, then

$$\sum_{p \in \mathscr{R}_{m-k}} \prod_{\ell = k+1}^{m} \|\hat{H}_\ell(n + D^k p)\|^2 \leqslant 1. \tag{2.21}$$

Note that it suffices to establish (2.21) for the collections $\mathscr{R}_k$, $k \geqslant 1$, that satisfy

$$\mathscr{R}_{k+1} = \mathscr{R}_k + D^k \mathscr{R}_1, \qquad k \geqslant 1. \tag{2.22}$$

For $n \in \mathbb{Z}^s$ and $m \geqslant k+1$, (2.22) yields

$$\begin{aligned} &\sum_{p \in \mathscr{R}_{m-k+1}} \prod_{\ell = k+1}^{m+1} \|\hat{H}_\ell(n + D^k p)\|^2 \\ &\quad = \sum_{p \in \mathscr{R}_{m-k}} \sum_{v \in \mathscr{R}_1} \prod_{\ell = k+1}^{m+1} \|\hat{H}_\ell(n + D^k(p + D^{m-k} v))\|^2 \\ &\quad = \sum_{p \in \mathscr{R}_{m-k}} \prod_{\ell = k+1}^{m} \|\hat{H}_\ell(n + D^k p)\|^2 \left(\sum_{v \in \mathscr{R}_1} \|\hat{H}_{m+1}((n + D^k p) + D^m v)\|^2 \right). \end{aligned}$$

It then follows from (2.20) that

$$\begin{aligned} &\sum_{p \in \mathscr{R}_{m-k+1}} \prod_{\ell = k+1}^{m+1} \|\hat{H}_\ell(n + D^k p)\|^2 \\ &\qquad \leqslant \sum_{p \in \mathscr{R}_{m-k}} \prod_{\ell = k+1}^{m} \|\hat{H}_\ell(n + D^k p)\|^2, \qquad m \geqslant k+1. \end{aligned} \tag{2.23}$$

Applying (2.23) repeatedly and using (2.20), we obtain

$$\sum_{p \in \mathscr{R}_{m-k+1}} \prod_{\ell = k+1}^{m+1} \|\hat{H}_\ell(n + D^k p)\|^2 \leqslant \sum_{p \in \mathscr{R}_1} \|\hat{H}_{k+1}(n + D^k p)\|^2 \leqslant 1.$$

It also follows from (2.20) that $\|\hat{H}_{k+1}(n)\|^2 \leqslant 1$ for all $n \in \mathbb{Z}^s$. Thus for $m \geqslant k+1$, $p \in \mathcal{R}_{m-k}$ and $n \in \mathbb{Z}^s$,

$$\prod_{\ell=k+1}^{\infty} \|\hat{H}_\ell(n+D^k p)\|^2 \leqslant \prod_{\ell=k+1}^{m} \|\hat{H}_\ell(n+D^k p)\|^2. \tag{2.24}$$

Since a_k is defined by (2.17), it follows from (2.16), (2.24), and (2.21) that

$$\sum_{p \in \mathcal{R}_{m-k}} \|a_k(n+D^k p)\|^2 \leqslant K^2, \qquad m \geqslant k+1, \tag{2.25}$$

where $\mathcal{R}_{m-k}$ is any full collection of coset representatives of $\mathbb{Z}^s/D^{m-k}\mathbb{Z}^s$.

Now, for a fixed positive number N, consider $B(N) := \{p \in \mathbb{Z}^s \colon \|p\| \leqslant N\}$. Then there exists a sufficiently large positive integer m such that the elements in $B(N)$ lie in different cosets of $\mathbb{Z}^s/D^{m-k}\mathbb{Z}^s$. Indeed, suppose that for every $m \geqslant k+1$, there exist $p_1, p_2 \in B(N)$, $p_1 \neq p_2$, that lie in the same coset of $\mathbb{Z}^s/D^{m-k}\mathbb{Z}^s$. Then we may write $p_1 - p_2 = D^{m-k}q$ for some $q \in \mathbb{Z}^s \backslash \{0\}$. Consequently,

$$\|D^{m-k}q\| \leqslant \|p_1\| + \|p_2\| \leqslant 2N.$$

On the other hand, since $q \neq 0$, we have

$$1 \leqslant \|q\| \leqslant \|D^{-(m-k)}\| \, \|D^{m-k}q\|.$$

Hence, for every $m \geqslant k+1$,

$$\frac{1}{\|D^{-(m-k)}\|} \leqslant 2N. \tag{2.26}$$

Since all the eigenvalues of D^{-1} lie inside the unit circle, (2.26) leads to a contradiction as $m \to \infty$.

With m such that the elements in $B(N)$ lie in different cosets of $\mathbb{Z}^s/D^{m-k}\mathbb{Z}^s$, choose $\mathcal{R}_{m-k}$ to be a full collection of coset representatives of $\mathbb{Z}^s/D^{m-k}\mathbb{Z}^s$ which contains $B(N)$. Then (2.25) implies that

$$\sum_{p \in B(N)} \|a_k(n+D^k p)\|^2 \leqslant \sum_{p \in \mathcal{R}_{m-k}} \|a_k(n+D^k p)\|^2 \leqslant K^2.$$

Taking limits as $N \to \infty$ gives (2.19) and completes the proof of the theorem. ∎

In view of Theorem 2.3, if $\hat{H}_{k+1}$ satisfies (2.18), and ϕ_k is a sequence of 2π-periodic vector functions defined by

$$\hat{\phi}_k(n) := a_k(n), \qquad n \in \mathbb{Z}^s,$$

then ϕ_k is an L^2-solution of the periodic matrix refinement equation (1.9).

For the nonperiodic case of $L^2(\mathbb{R}^s)^r$, solution of the matrix refinement equation (1.10) with finitely supported masks are often studied in terms of the cascade algorithm and transition operators. In [13], general results on the convergence of nonstationary vector cascade algorithms were obtained. It should be mentioned that based on the approach in [13], transition operators corresponding to periodic analogues of finitely supported masks can be defined and used to characterize solutions of the periodic matrix refinement equation (1.9).

3. LINEAR INDEPENDENCE OF SOLUTIONS OF PERIODIC REFINEMENT EQUATIONS

Fix $k \geqslant 0$. To facilitate the discussion on the linear independence of the shifts of scaling functions, we need to choose suitable coset representatives of $\mathbb{Z}^s/M^k\mathbb{Z}^s$ and $\mathbb{Z}^s/D^k\mathbb{Z}^s$. To this end, let $\{e_1, ..., e_s\}$ be the standard basis of the free abelian group $\mathbb{Z}^s$, and let $f_1, ..., f_s$ and $g_1, ..., g_s$ be the generators of $M^k\mathbb{Z}^s$ and $D^k\mathbb{Z}^s$ respectively, where $f_1, ..., f_s$ and $g_1, ..., g_s$ are the column vectors of M^k and D^k respectively. Then

$$f_j = \sum_{v=1}^{s} d_{jv} e_v, \qquad g_j = \sum_{v=1}^{s} m_{jv} e_v, \tag{3.1}$$

where d_{jv} and m_{jv} denote the (j, v)-entry of D^k and M^k respectively.

Now, there exist invertible matrices P and Q in $M_s(\mathbb{Z})$, the ring of all $s \times s$ matrices with integer entries, such that

$$QD^kP^{-1} = \operatorname{diag}(n_1, ..., n_s) = (P^{-1})^T M^k Q^T, \tag{3.2}$$

where $n_1, ..., n_s$ are positive integers (see [16, Theorem 3.8]). Note that

$$d^k = |\det(M^k)| = |\det(D^k)| = n_1 n_2 \cdots n_s,$$

where d is defined by (1.2). Write $P = (p_{ij})$, $Q = (q_{ij})$, $P^{-1} = (\tilde{p}_{ij})$, and $Q^{-1} = (\tilde{q}_{ij})$. If we set

$$e'_i = \sum_{j=1}^{s} p_{ij} e_j, \qquad f'_i = \sum_{j=1}^{s} q_{ij} f_j, \qquad e''_i = \sum_{j=1}^{s} \tilde{q}_{ji} e_j, \qquad g'_i = \sum_{j=1}^{s} \tilde{p}_{ji} g_j,$$

then it follows from (3.1) and (3.2) that $f'_i = n_i e'_i$ and $g'_i = n_i e''_i$, $i = 1, 2, ..., s$. Furthermore, $\{e'_1, ..., e'_s\}$ and $\{e''_1, ..., e''_s\}$ are bases of $\mathbb{Z}^s$, and $\{f'_1, ..., f'_s\}$ and $\{g'_s, ..., g'_s\}$ are also sets of generators of $M^k\mathbb{Z}^s$ and $D^k\mathbb{Z}^s$ respectively. Hence, we may choose the sets $\mathscr{L}_k$ and $\mathscr{R}_k$ of all coset representatives of $\mathbb{Z}^s/M^k\mathbb{Z}^s$ and $\mathbb{Z}^s/D^k\mathbb{Z}^s$ respectively to be

$$\mathscr{L}_k = \{m_1 e'_1 + \cdots + m_s e'_s : 0 \leqslant m_i < n_i,\ i = 1, 2, ..., s\}$$

and

$$\mathscr{R}_k = \{\mu_1 e''_1 + \cdots + \mu_s e''_s : 0 \leqslant \mu_i < n_i,\ i = 1, 2, ..., s\}.$$

We order $\mathscr{L}_k$ in such a way that

$$m_1 e'_1 + \cdots + m_s e'_s < r_1 e'_1 + \cdots + r_s e'_s$$

if and only if there exists a least integer i, $i = 1, 2, ..., s$, for which $m_i < r_i$. Write $\mathscr{L}_k = \{\ell_1, ..., \ell_{d^k}\}$, where $\ell_1 < \cdots < \ell_{d^k}$.

Suppose that $\phi_k = (\phi_k^1, ..., \phi_k^r)^T$ is an L^2-solution of the periodic matrix refinement equation (1.9). The linear independence of the set $S_k := \{T_k^{\ell}\phi_k^m : m = 1, 2, ..., r,\ \ell \in \mathscr{L}_k\}$ is equivalent to the invertibility of its Gram matrix $\mathscr{G}_k$, which can be expressed as $\mathscr{G}_k := (\Phi_{m\mu})_{m,\mu=1}^r$, where

$$\begin{aligned}\Phi_{m\mu} &:= (\langle T_k^{\ell_p}\phi_k^m, T_k^{\ell_q}\phi_k^\mu\rangle)_{p,q=1}^{d^k} \\ &= (\langle \phi_k^m, T_k^{\ell_q - \ell_p}\phi_k^\mu\rangle)_{p,q=1}^{d^k}, \qquad m, \mu = 1, 2, ..., r,\end{aligned}$$

are circulant matrices of level s and type $(n_1, ..., n_s)$ (see [11, pp. 184–188]). Note that the matrix $\mathscr{G}_k$ is Hermitian and positive semi-definite. By [11, Theorem 5.8.4], we have

$$\Phi_{m\mu} = F^* \left(\sum_{\gamma_1=0}^{n_1-1} \cdots \sum_{\gamma_s=0}^{n_s-1} \langle \phi_k^m, \phi_k^\mu(\cdot - 2\pi M^{-k}(\gamma_1 e'_1 + \cdots + \gamma_s e'_s))\rangle\, \Omega^\gamma \right) F.$$

Here, $F = F_{n_1} \otimes \cdots \otimes F_{n_s}$ is the Kronecker product of the Fourier matrices F_{n_v}, $v = 1, 2, ..., s$ (see [11] for the definitions), and $\Omega^\gamma = \Omega_{n_1}^{\gamma_1} \otimes \cdots \otimes \Omega_{n_s}^{\gamma_s}$, where

$$\Omega_{n_v} = \operatorname{diag}(1, \omega_{n_v}, \omega_{n_v}^2, ..., \omega_{n_v}^{n_v - 1}), \qquad \omega_{n_v} = \exp(2\pi i/n_v).$$

Consequently, we see that a typical eigenvalue of $\Phi_{m\mu}$ is

$$\sum_{\gamma_1=0}^{n_1-1} \cdots \sum_{\gamma_s=0}^{n_s-1} \langle \phi_k^m, \phi_k^\mu(\cdot - 2\pi M^{-k}(\gamma_1 e_1' + \cdots + \gamma_s e_s')) \rangle \, \omega_{n_1}^{\gamma_1 \alpha_1} \cdots \omega_{n_s}^{\gamma_s \alpha_s}$$

$$= \sum_{n \in \mathbb{Z}^s} \hat{\phi}_k^m(n) \overline{\hat{\phi}_k^\mu(n)} \sum_{\gamma_1=0}^{n_1-1} \cdots \sum_{\gamma_s=0}^{n_s-1} \exp(in \cdot (2\pi M^{-k}(\gamma_1 e_1' + \cdots + \gamma_s e_s')))$$

$$\times \omega_{n_1}^{\gamma_1 \alpha_1} \cdots \omega_{n_s}^{\gamma_s \alpha_s}, \tag{3.3}$$

where $\alpha_v = 0, 1, ..., n_v - 1$, $v = 1, 2, ..., s$. Using (3.2), (3.3) and the chosen collections of coset representatives $\mathscr{L}_k$ and $\mathscr{R}_k$, the eigenvalues of $\Phi_{m\mu}$ can be written as

$$\lambda_{m\mu}^j := d^k \sum_{p \in \mathbb{Z}^s} \hat{\phi}_k^m(j + D^k p) \overline{\hat{\phi}_k^\mu(j + D^k p)}, \qquad j \in \mathscr{R}_k. \tag{3.4}$$

Let us define *polyphase splines* $v_{k,j}^m$ by

$$v_{k,j}^m(x) := \sum_{p \in \mathbb{Z}^s} \hat{\phi}_k^m(j + D^k p) \, e^{i(j + D^k p) \cdot x}, \qquad x \in \mathbb{R}^s, \tag{3.5}$$

for $m = 1, 2, ..., r$, $j \in \mathscr{R}_k$. Then it is easy to see that

$$\langle v_{k,j}^m, v_{k,j}^\mu \rangle = \sum_{p \in \mathbb{Z}^s} \hat{\phi}_k^m(j + D^k p) \overline{\hat{\phi}_k^\mu(j + D^k p)} = \frac{\lambda_{m\mu}^j}{d^k}. \tag{3.6}$$

Note that for every $p \in \mathbb{Z}^s$, $j \in \mathscr{R}_k$, we denote $v_{k,j+D^k p}^m := v_{k,j}^m$.

The polyphase splines are extensions of the orthogonal splines (for the scalar case of $r = 1$) introduced in [18]. The orthogonal splines are always orthogonal to each other whereas the polyphase splines need not be so. However, the polyphase splines can still be used to characterize the linear independence of $S_k = \{T_k^\ell \phi_k^m : m = 1, 2, ..., r, \ell \in \mathscr{L}_k\}$.

Now, for each $j \in \mathscr{R}_k$, let

$$M_k(j) := (\langle v_{k,j}^m, v_{k,j}^\mu \rangle)_{m,\mu=1}^r = \frac{1}{d^k} (\lambda_{m\mu}^j)_{m,\mu=1}^r. \tag{3.7}$$

Note that $M_k(j)$ is positive semi-definite, and by (3.6), we have

$$M_k(j) = \sum_{p \in \mathbb{Z}^s} \hat{\phi}_k(j + D^k p) \, \hat{\phi}_k(j + D^k p)^*$$

$$= \sum_{\ell \in \mathscr{R}_1} \sum_{p \in \mathbb{Z}^s} \hat{\phi}_k(j + D^k \ell + D^{k+1} p) \, \hat{\phi}_k(j + D^k \ell + D^{k+1} p)^*. \tag{3.8}$$

Since $\hat{\phi}_k$ satisfies (1.9), we deduce from (3.8) that

$$M_k(j) = \sum_{\ell \in \mathscr{R}_1} \hat{H}_{k+1}(j + D^k \ell)\, M_{k+1}(j + D^k \ell)\, \hat{H}_{k+1}(j + D^k \ell)^*. \quad (3.9)$$

PROPOSITION 3.1. *The following are equivalent.*

(i) S_k *is linearly independent.*

(ii) $\det(\mathscr{G}_k) > 0$.

(iii) *For each* $j \in \mathscr{R}_k$, $\det(M_k(j)) > 0$.

(iv) *For each* $j \in \mathscr{R}_k$, $\{v^m_{k,j} : m = 1, 2, ..., r\}$ *is linearly independent.*

Proposition 3.1 follows from the following lemma.

LEMMA 3.1. *Let* $C := (C_{ij})^r_{i,j=1}$ *be a block matrix of size* $nr \times nr$, *where* C_{ij}, $i, j = 1, 2, ..., r$, *are* $n \times n$ *circulant matrices of level* s. *Let* λ^ℓ_{ij}, $\ell = 1, 2, ..., n$, *be the eigenvalues of* C_{ij}. *Set*

$$\Lambda(\ell) := (\lambda^\ell_{ij})^r_{i,j=1} \qquad \text{and} \qquad \Lambda := \operatorname{diag}(\Lambda(1), ..., \Lambda(n)).$$

Then there exists a unitary matrix U *such that*

$$C = U^* \Lambda U. \quad (3.10)$$

Proof. For $i, j = 1, 2, ..., r$, let

$$B_{ij} := \operatorname{diag}(\lambda^1_{ij}, ..., \lambda^n_{ij})$$

and $B := (B_{ij})^r_{i,j=1}$. Then $C_{ij} = F^* B_{ij} F$ for some unitary matrix F (a Kronecker product of s Fourier matrices) (see [11]). It follows that

$$C = \operatorname{diag}(F^*, ..., F^*)\, B \operatorname{diag}(F, ..., F), \quad (3.11)$$

where $\operatorname{diag}(F, ..., F)$ is a block diagonal matrix with r diagonal blocks each of which is of order n. Observe that Λ can be obtained from B by interchanging rows and columns. Thus, there exists an orthogonal matrix P such that $B = P^* \Lambda P$. Setting $U = P \operatorname{diag}(F, ..., F)$ in (3.11) leads to (3.10). ∎

The proof of Proposition 3.1 also leads to equivalent conditions for $S_k = \{T^\ell_k \phi^m_k : m = 1, 2, ..., r, \ell \in \mathscr{L}_k\}$ to be an orthonormal set.

COROLLARY 3.1. *The following are equivalent.*

(i) S_k *is an orthonormal set.*

(ii) $\mathscr{G}_k = I_{rd^k}$, *the* $rd^k \times rd^k$ *identity matrix.*

(iii) *For each* $j \in \mathcal{R}_k$, $M_k(j) = (1/d^k)\, I_r$.

(iv) *For each* $j \in \mathcal{R}_k$, $\{v_{k,j}^m : m = 1, 2, ..., r\}$ *is an orthogonal set, and* $\|v_{k,j}^m\|^2 = 1/d^k$, $m = 1, 2, ..., r$.

Suppose now that the set $\{T_k^\ell \phi_k^m : m = 1, 2, ..., r, \ell \in \mathcal{L}_k\}$ forms a basis of V_k. Then

$$V_k = \left\{ f \in L^2([0, 2\pi)^s) : f = \sum_{m=1}^{r} \sum_{\ell \in \mathcal{L}_k} \alpha_m(\ell)\, T_k^\ell \phi_k^m,\ \alpha_1, ..., \alpha_r \in \mathcal{S}(M^k) \right\}.$$

Equivalently,

$$V_k = \left\{ f \in L^2([0, 2\pi)^s) : \hat{f}(n) = \sum_{m=1}^{r} \hat{\alpha}_m(n)\, \hat{\phi}_k^m(n),\ \hat{\alpha}_1, ..., \hat{\alpha}_r \in \mathcal{S}(D^k) \right\}. \qquad (3.12)$$

It turns out that the polyphase splines $v_{k,j}^m$, $m = 1, 2, ..., r$, $j \in \mathcal{R}_k$, form an alternative basis of V_k, and such a basis will greatly facilitate the construction of multiwavelets in the next section.

PROPOSITION 3.2. *Suppose that* ϕ_k, $k \geq 0$, *is an* L^2*-solution of the periodic matrix refinement equation* (1.9). *Then for* $k \geq 0$,

$$S_k = \{T_k^\ell \phi_k^m : m = 1, 2, ..., r, \ell \in \mathcal{L}_k\}$$

is a basis of V_k *if and only if*

$$\{v_{k,j}^m : m = 1, 2, ..., r,\ j \in \mathcal{R}_k\}$$

is also a basis of V_k, *where* $v_{k,j}^m$ *are as defined in* (3.5). *Furthermore,*

$$v_{k,j} = \sum_{\ell \in \mathcal{R}_1} \hat{H}_{k+1}(j + D^k \ell)\, v_{k+1,\, j + D^k \ell}, \qquad j \in \mathcal{R}_k, \qquad (3.13)$$

where $v_{k,j} = (v_{k,j}^1, ..., v_{k,j}^r)^T$.

Proof. Suppose that $\{v_{k,j}^m : m = 1, 2, ..., r, j \in \mathcal{R}_k\}$ is a basis of V_k. Then $\{v_{k,j}^m : m = 1, 2, ..., r\}$ is linearly independent for each $j \in \mathcal{R}_k$. It follows from Proposition 3.1 that S_k is linearly independent, and hence a basis of V_k.

Conversely, suppose that S_k is a basis of V_k. For each $m = 1, 2, ..., r$ and $j \in \mathcal{R}_k$, since

$$v_{k,j}^m(x) = \sum_{p \in \mathbb{Z}^s} \hat{\phi}_k^m(j + D^k p)\, e^{i(j + D^k p) \cdot x},$$

we have

$$\hat{v}^m_{k,j}(n) = \begin{cases} \hat{\phi}^m_k(j + D^k p), & \text{if} \quad n = j + D^k p \text{ for some } p \in \mathbb{Z}^s, \\ 0, & \text{otherwise.} \end{cases} \tag{3.14}$$

Thus, if we define $\hat{\alpha}_1, ..., \hat{\alpha}_r \in \mathscr{S}(D^k)$ by $\hat{\alpha}_\mu \equiv 0$ for $\mu \neq m$, and

$$\hat{\alpha}_m(\ell) = \begin{cases} 1, & \text{if} \quad \ell = j, \\ 0, & \text{if} \quad \ell \neq j, \end{cases}$$

then $\hat{v}^m_{k,j}(n) = \sum_{\mu=1}^r \hat{\alpha}_\mu(n)\, \hat{\phi}^\mu_k(n)$ for all $n \in \mathbb{Z}^s$. By (3.12), this shows that $v^m_{k,j} \in V_k$ for all $m = 1, 2, ..., r$ and $j \in \mathscr{R}_k$.

Now, to prove that the set $\{v^m_{k,j} \colon m = 1, 2, ..., r,\ j \in \mathscr{R}_k\}$ is a basis of V_k, it suffices to show that it is linearly independent, since $\dim V_k = r\,|\mathscr{R}_k|$. By Parseval's identity and (3.14), we have

$$\langle v^m_{k,j}, v^\mu_{k,\ell} \rangle = \sum_{n \in \mathbb{Z}^s} \hat{v}^m_{k,j}(n)\, \overline{\hat{v}^\mu_{k,\ell}(n)} = 0 \qquad \text{if} \quad j \neq \ell, \tag{3.15}$$

for all $m, \mu = 1, 2, ..., r$. Suppose that

$$\sum_{\mu=1}^{r} \sum_{j \in \mathscr{R}_k} c_{\mu,j} v^\mu_{k,j} = 0, \tag{3.16}$$

where $c_{\mu,j} \in \mathbb{C}$. Fix $\ell \in \mathscr{R}_k$. Then, by (3.15), for each $m = 1, 2, ..., r$, taking inner product on both sides of (3.16) with $v^m_{k,\ell}$ gives

$$\sum_{\mu=1}^{r} \bar{c}_{\mu,\ell} \langle v^m_{k,\ell}, v^\mu_{k,\ell} \rangle = 0.$$

Hence we obtain the matrix equation

$$M_k(\ell)(\bar{c}_{1,\ell}, ..., \bar{c}_{r,\ell})^T = (0, ..., 0)^T,$$

where $M_k(\ell) = (\langle v^m_{k,\ell}, v^\mu_{k,\ell} \rangle)^r_{m,\mu=1}$. By the hypothesis and Proposition 3.1, $M_k(\ell)$ is invertible. It follows that $c_{1,\ell} = \cdots = c_{r,\ell} = 0$. Since ℓ is arbitrary in $\mathscr{R}_k$, this completes the proof for the first part of the proposition.

The derivation of (3.13) is similar to that of [18, Proposition 5.1]. ▮

We have identified conditions for the scaling functions ϕ^m_k, $k \geqslant 0$, $m = 1, 2, ..., r$, to generate subspaces $\{V_k \colon k \geqslant 0\}$ satisfying MR1 and MR2. For $\{V_k \colon k \geqslant 0\}$ to form a periodic multiresolution, $\bigcup_{k \geqslant 0} V_k$ must be dense in $L^2([0, 2\pi)^s)$ (MR3). The following theorem is a periodic analogue of a theorem in [1], and it leads to a characterization of MR3. For the special case of $s = 1$, the result was established independently in [18] and [25].

THEOREM 3.1. *Let V be a subspace of $L^2([0, 2\pi)^s)$ such that for all $k=0, 1, ..., \ell \in \mathscr{L}_k$,*

$$f \in V \Rightarrow f(\cdot - 2\pi M^{-k}\ell) \in V. \tag{3.17}$$

Then V is dense in $L^2([0, 2\pi)^s)$ if and only if the set

$$\Omega := \bigcap_{f \in V} \{n \in \mathbb{Z}^s \colon \hat{f}(n) = 0\} \tag{3.18}$$

is empty.

Proof. Let V be dense in $L^2([0, 2\pi)^s)$. Then $V^\perp = \{0\}$. If the set Ω defined in (3.18) is not empty, then there exists $n_0 \in \mathbb{Z}^s$ such that $\hat{f}(n_0) = 0$ for all $f \in V$. Thus

$$\langle f, e^{in_0 \cdot} \rangle = 0 \qquad \text{for all} \quad f \in V.$$

In other words, the function $e^{in_0 \cdot}$ lies inside $V^\perp$, which is a contradiction.

Conversely, suppose that Ω is empty. To prove that V is dense in $L^2([0, 2\pi)^s)$, it suffices to show that if $g \in L^2([0, 2\pi)^s)$ is orthogonal to V, then $g = 0$. Let $g \in L^2([0, 2\pi)^s)$ satisfy

$$\langle f, g \rangle = 0 \qquad \text{for all} \quad f \in V.$$

Then it follows from (3.17) that for all $k = 0, 1, ..., \ell \in \mathscr{L}_k$ and $f \in V$,

$$\langle f(\cdot - 2\pi M^{-k}\ell), g \rangle = 0. \tag{3.19}$$

By Parseval's identity,

$$\langle f(\cdot - 2\pi M^{-k}\ell), g \rangle = \sum_{n \in \mathbb{Z}^s} \hat{f}(n)\, \overline{\hat{g}(n)}\, e^{-in \cdot (2\pi M^{-k}\ell)}.$$

Thus (3.19) yields

$$\sum_{j \in \mathscr{R}_k} \sum_{p \in \mathbb{Z}^s} \hat{f}(j + D^k p)\, \overline{\hat{g}(j + D^k p)}\, e^{-ij \cdot (2\pi M^{-k}\ell)} = 0, \qquad \ell \in \mathscr{L}_k.$$

Consequently, in view of the finite Fourier transform (1.4) and its inverse (1.5), we see that

$$\sum_{p \in \mathbb{Z}^s} \hat{f}(j + D^k p)\, \overline{\hat{g}(j + D^k p)} = 0, \tag{3.20}$$

for all $k \geqslant 0$, $j \in \mathscr{R}_k$.

Now, the series $\sum_{n \in \mathbb{Z}^s} \hat{f}(n)\,\overline{\hat{g}(n)}$ is absolutely convergent since $f, g \in L^2([0, 2\pi)^s)$. Thus for any $\varepsilon > 0$, there exists a positive number N such that $\sum_{\|m\| > N} |\hat{f}(m)\,\overline{\hat{g}(m)}| < \varepsilon$. Fix $n \in \mathbb{Z}^s$. As in the proof of Theorem 2.3, due to the fact that all the eigenvalues of D^{-1} lie inside the unit circle, there exists a positive integer K such that for $k > K$ and $p \in \mathbb{Z}^s \backslash \{0\}$,

$$\|D^k p\| > N + \|n\|.$$

Consequently, choosing $k > K$, (3.20) yields

$$\begin{aligned} |\hat{f}(n)\,\overline{\hat{g}(n)}| &= \left| \sum_{p \in \mathbb{Z}^s \backslash \{0\}} \hat{f}(n + D^k p)\,\overline{\hat{g}(n + D^k p)} \right| \\ &\leqslant \sum_{\|m\| > N} |\hat{f}(m)\,\overline{\hat{g}(m)}| < \varepsilon. \end{aligned}$$

Hence,

$$\hat{f}(n)\,\overline{\hat{g}(n)} = 0 \qquad \text{for all} \quad n \in \mathbb{Z}^s, \quad f \in V. \tag{3.21}$$

Using the assumption that the set Ω is empty, we conclude from (3.21) that $\hat{g}(n) = 0$ for all $n \in \mathbb{Z}^s$. Thus $g = 0$ and the proof is complete. ∎

Corollary 3.2. *If $\{V_k : k \geqslant 0\}$ is a sequence of subspaces of $L^2([0, 2\pi)^s)$ satisfying* MR1 *and* MR2, *then $\bigcup_{k \geqslant 0} V_k$ is dense in $L^2([0, 2\pi)^s)$ if and only if the set*

$$\{n \in \mathbb{Z}^s : \hat{\phi}_k^m(n) = 0 \textit{ for all } k \geqslant 0, m = 1, 2, ..., r\}$$

is empty.

Proof. Note that the space $V := \bigcup_{k \geqslant 0} V_k$ satisfies the condition (3.17) in Theorem 3.1. By the characterization (3.12) of V_k, we observe that

$$\bigcap_{f \in V} \{n \in \mathbb{Z}^s : \hat{f}(n) = 0\} = \{n \in \mathbb{Z}^s : \hat{\phi}_k^m(n) = 0 \text{ for all } k \geqslant 0, m = 1, 2, ..., r\}.$$

Then the result follows immediately from Theorem 3.1. ∎

4. CONSTRUCTION OF MULTIDIMENSIONAL PERIODIC MULTIWAVELETS

Throughout this section, we assume that the solution of the periodic matrix refinement equation (1.9) $a_k \in \ell^2(\mathbb{Z}^s)^r$, $k \geqslant 0$, and that the sequence ϕ_k defined by $\hat{\phi}_k := a_k$ generates a multiresolution $\{V_k : k \geqslant 0\}$ of $L^2([0, 2\pi)^s)$.

Let $v_{k,j}$, $j \in \mathscr{R}_k$, be the corresponding polyphase splines, and let W_k be the orthogonal complement of V_k in V_{k+1}. Since $\dim V_k = rd^k$ where d is defined by (1.2), we have $\dim W_k = rd^k(d-1)$. Our strategy here is to first construct a polyphase spline basis of W_k and then use it to obtain a multiwavelet basis of W_k.

Consider another sequence of vectors $b_k \in \ell^2(\mathbb{Z}^s)^{r(d-1)}$, $k \geqslant 0$, such that

$$b_k(n) = \hat{G}_{k+1}(n)\, a_{k+1}(n), \qquad n \in \mathbb{Z}^s, \tag{4.1}$$

where $\hat{G}_{k+1} \in \mathscr{S}(D^{k+1})^{r(d-1)\times r}$, the class of periodic sequences of $r(d-1)\times r$ complex matrices of period D^{k+1}. Define polyphase splines

$$u^m_{k,j}(x) := \sum_{p \in \mathbb{Z}^s} b^m_k(j + D^k p)\, e^{i(j + D^k p)\cdot x}, \qquad x \in \mathbb{R}^s, \tag{4.2}$$

for $k \geqslant 0$, $j \in \mathscr{R}_k$, $m = 1, 2, ..., r(d-1)$. Note that for every $p \in \mathbb{Z}^s$, $j \in \mathscr{R}_k$, we denote $u^m_{k,j+D^k p} := u^m_{k,j}$. Let

$$\psi^m_k := \sum_{j \in \mathscr{R}_k} u^m_{k,j}. \tag{4.3}$$

PROPOSITION 4.1. *The following are equivalent.*

(i) *For each* $j \in \mathscr{R}_k$, $\det((\sum_{p\in\mathbb{Z}^s} b^m_k(j + D^k p)\, \overline{b^\mu_k(j + D^k p)})^{r(d-1)}_{m,\mu=1}) > 0$.

(ii) *For each* $j \in \mathscr{R}_k$, $\{u^m_{k,j}: m = 1, 2, ..., r(d-1)\}$ *is linearly independent.*

(iii) $\{T^\ell_k \psi^m_k: m = 1, 2, ..., r(d-1), \ell \in \mathscr{L}_k\}$ *is linearly independent.*

(iv) $\{u^m_{k,j}: m = 1, 2, ..., r(d-1), j \in \mathscr{R}_k\}$ *is linearly independent.*

Proof. Similar to the proofs of Propositions 3.1 and 3.2. ∎

For each $k \geqslant 0$, let

$$N_k(j) := (\langle u^m_{k,j}, u^\mu_{k,j}\rangle)^{r(d-1)}_{m,\mu=1}, \qquad j \in \mathscr{R}_k. \tag{4.4}$$

Then $N_k(j)$ is positive semi-definite, and (4.2) implies that

$$\begin{aligned} N_k(j) &= \left(\sum_{p\in\mathbb{Z}^s} b^m_k(j + D^k p)\, \overline{b^\mu_k(j + D^k p)}\right)^{r(d-1)}_{m,\mu=1} \\ &= \sum_{p\in\mathbb{Z}^s} b_k(j + D^k p)\, b_k(j + D^k p)^*. \end{aligned} \tag{4.5}$$

As in the derivation of (3.9), we deduce from (4.1) and (4.5) that

$$N_k(j) = \sum_{\ell \in \mathscr{R}_1} \hat{G}_{k+1}(j + D^k \ell)\, M_{k+1}(j + D^k \ell)\, \hat{G}_{k+1}(j + D^k \ell)^*, \tag{4.6}$$

where $M_k(j)$ are as defined in (3.7). Hence, Condition (i) of Proposition 4.1 is equivalent to

$$\det\left(\sum_{\ell\in\mathscr{R}_1} \hat{G}_{k+1}(j+D^k\ell)\, M_{k+1}(j+D^k\ell)\, \hat{G}_{k+1}(j+D^k\ell)^*\right)>0, \quad (4.7)$$

for all $j\in\mathscr{R}_k$.

Analogous to (3.13) of Proposition 3.2, we note that (4.1) and (4.2) imply that

$$u_{k,j}=\sum_{\ell\in\mathscr{R}_1} \hat{G}_{k+1}(j+D^k\ell)\, v_{k+1,\, j+D^k\ell}, \qquad j\in\mathscr{R}_k, \quad (4.8)$$

where $u_{k,j}=(u^1_{k,j}, ..., u^{r(d-1)}_{k,j})^T$. As a consequence, we have $u^m_{k,j}\in V_{k+1}$ for all $m=1, 2, ..., r(d-1)$, $j\in\mathscr{R}_k$. The following proposition gives a necessary and sufficient condition for $u^m_{k,j}$ to be in W_k.

PROPOSITION 4.2. *The functions $u^m_{k,j}$ lie in W_k for all $m=1, 2, ..., r(d-1)$ and $j\in\mathscr{R}_k$ if and only if*

$$\sum_{\ell\in\mathscr{R}_1} \hat{G}_{k+1}(j+D^k\ell)\, M_{k+1}(j+D^k\ell)\, \hat{H}_{k+1}(j+D^k\ell)^*=0, \quad (4.9)$$

for all $j\in\mathscr{R}_k$.

Proof. For each $m=1, 2, ..., r(d-1)$, $j\in\mathscr{R}_k$, since $u^m_{k,j}\in V_{k+1}$ and W_k is the orthogonal complement of V_k in V_{k+1}, we see that $u^m_{k,j}\in W_k$ if and only if $\langle u^m_{k,j}, v^\mu_{k,\ell}\rangle=0$ for all $\mu=1, 2, ..., r$, $\ell\in\mathscr{R}_k$. It is clear that for all μ, $\langle u^m_{k,j}, v^\mu_{k,\ell}\rangle=0$ if $j\neq\ell$. Hence, $u^m_{k,j}\in W_k$ for all $m=1, 2, ..., r(d-1)$ and $j\in\mathscr{R}_k$ if and only if

$$(\langle u^m_{k,j}, v^\mu_{k,j}\rangle)^{r(d-1),\, r}_{m,\mu=1}=0 \qquad \text{for all} \quad j\in\mathscr{R}_k. \quad (4.10)$$

Now, by (3.13), (3.15), and (4.8), we deduce that (4.10) is precisely (4.9), and this concludes the proof of the proposition. ∎

With the given sequence of vectors $a_k\in\ell^2(\mathbb{Z}^s)^r$ and the associated $v_{k,j}$ and $\hat{H}_{k+1}\in\mathscr{S}(D^{k+1})^{r\times r}$, Propositions 4.1 and 4.2 show that the construction of a polyphase spline basis (hence a wavelet basis via (4.3)) for W_k amounts to finding $\hat{G}_{k+1}\in\mathscr{S}(D^{k+1})^{r(d-1)\times r}$ such that the conditions (4.7) and (4.9) are satisfied. Therefore, the wavelet construction problem can be formulated as follows:

PROBLEM. *For each* $k=0, 1, ...,$ *given that*

$$\det(M_k(j)) = \det\left(\sum_{\ell \in \mathcal{R}_1} \hat{H}_{k+1}(j+D^k\ell)\, M_{k+1}(j+D^k\ell)\, \hat{H}_{k+1}(j+D^k\ell)^*\right) > 0 \tag{4.11}$$

for all $j \in \mathcal{R}_k$, *can we find* $\hat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{r(d-1)\times r}$ *such that* (4.7) *and* (4.9) *hold for all* $j \in \mathcal{R}_k$?

Before proving that the answer to the above question is affirmative, we need the following lemma.

LEMMA 4.1. *Let* H *and* G *be* $p\times n$ *and* $q\times n$ *matrices respectively, where* $p+q=n$. *Suppose that the* $n\times n$ *matrix* $\left(\frac{H}{G}\right)$ *is invertible and* $HG^*=0$. *Then for any* $n\times n$ *positive definite Hermitian matrix* S, *the matrix* $\left(\frac{HS}{G}\right)$ *is invertible.*

Proof. Since S is a positive definite Hermitian matrix, there exists a unitary $n\times n$ matrix U such that $U^*SU=\Lambda$, where Λ is a diagonal matrix whose diagonal entries are positive real numbers. Therefore $S=U\Lambda U^*$. Note that

$$\left(\frac{H}{G}\right) U = \left(\frac{HU}{GU}\right)$$

is invertible, and $(HU)(GU)^* = HG^* = 0$. Thus, if we can show that $\left(\frac{HU\Lambda}{GU}\right)$ is invertible, then $\left(\frac{HS}{G}\right) = \left(\frac{HU\Lambda}{GU}\right) U^*$ will be invertible. We have therefore reduced the proof of the lemma to the case when S is a diagonal matrix whose diagonal entries are all positive real numbers.

Let

$$H = \begin{pmatrix} v_1 \\ \hline \vdots \\ \hline v_p \end{pmatrix}, \qquad G = \begin{pmatrix} u_1 \\ \hline \vdots \\ \hline u_q \end{pmatrix},$$

where v_i and u_j are $1\times n$ matrices. Note that $\{v_1, ..., v_p\}$ and $\{u_1, ..., u_q\}$ are linearly independent sets of vectors in $\mathbb{C}^n$. Let $S=\mathrm{diag}(\alpha_1, ..., \alpha_n)$, $\alpha_i>0$, and

$$HS = \begin{pmatrix} v_1' \\ \hline \vdots \\ \hline v_p' \end{pmatrix},$$

where v_i', $i=1, 2, ..., p$, are $1 \times n$ matrices. Then the set $\{v_1', ..., v_p'\}$ is again linearly independent in $\mathbb{C}^n$.

Suppose to the contrary that the matrix $(\frac{HS}{G})$ is not invertible. Then there exist $\lambda_1, ..., \lambda_p, \gamma_1, ..., \gamma_q \in \mathbb{C}$, with $\lambda_k \neq 0$ and $\gamma_\ell \neq 0$ for some $k=1, 2, ..., p$, $\ell=1, 2, ..., q$, such that

$$\lambda_1 v_1' + \cdots + \lambda_p v_p' + \gamma_1 u_1 + \cdots + \gamma_q u_q = 0. \tag{4.12}$$

Since $HG^* = 0$, we have $\langle v_i, u_j \rangle = 0$ for all $i=1, 2, ..., p$, $j=1, 2, ..., q$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{C}^n$. Hence, for each $i=1, 2, ..., p$, taking inner product on both sides of (4.12) with v_i gives

$$\sum_{j=1}^{p} \langle v_i, v_j' \rangle \bar{\lambda}_j = 0.$$

Consequently, we obtain the matrix equation

$$Ax = 0, \tag{4.13}$$

where $A = (\langle v_i, v_j' \rangle)_{i, j=1}^{p}$, $x = (\bar{\lambda}_1, ..., \bar{\lambda}_p)^T \neq 0$.

Now, we claim that A is invertible. To see this, we define row vectors $w_1, ..., w_p \in \mathbb{C}^n$ by

$$\begin{pmatrix} w_1 \\ \hline \vdots \\ \hline w_p \end{pmatrix} := H \operatorname{diag}(\sqrt{\alpha_1}, ..., \sqrt{\alpha_n}).$$

Then it can be easily checked that

$$\langle w_i, w_j \rangle = \langle v_i, v_j' \rangle \qquad \text{for all} \quad i, j = 1, 2, ..., p.$$

Therefore A is the Gram matrix of the linearly independent set $\{w_1, ..., w_p\}$, and it follows that A is invertible. Hence, (4.13) implies that $x=0$, which is a contradiction. ▮

THEOREM 4.1. *Suppose that for each $k \geqslant 0$ and $j \in \mathcal{R}_k$, $\det(M_k(j)) > 0$. Then there exist $\hat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{r(d-1) \times r}$ such that the conditions (4.7) and (4.9) are satisfied for all $k \geqslant 0$ and $j \in \mathcal{R}_k$.*

Proof. Fix $k \geqslant 0$, $j \in \mathcal{R}_k$. Let A be the $rd \times r$ matrix defined by

$$A := (\hat{H}_{k+1}(j + D^k \ell_1)\, M_{k+1}(j + D^k \ell_1) | \cdots |$$
$$\hat{H}_{k+1}(j + D^k \ell_d)\, M_{k+1}(j + D^k \ell_d))^*,$$

where $\ell_1, ..., \ell_d$ denote all the elements of $\mathscr{R}_1$. By the hypothesis, the matrix

$$(\hat{H}_{k+1}(j+D^k\ell_1)|\cdots|\hat{H}_{k+1}(j+D^k\ell_d))\, A$$

is invertible, so A has rank r. Thus, we can find an invertible $rd\times rd$ matrix Q (a product of elementary matrices) such that

$$QA=\left(\frac{T_r}{0}\right),$$

where T_r denotes an $r\times r$ upper-triangular matrix of rank r. Write $Q=(\frac{Q_1}{Q_2})$, where Q_1 and Q_2 are matrices of sizes $r\times rd$ and $r(d-1)\times rd$ respectively. Note that Q_2 is of rank $r(d-1)$, and $Q_2A=0$. Hence, if we define $\hat{G}_{k+1}(j+D^k\ell)$, $\ell\in\mathscr{R}_1$, by

$$Q_2=(\hat{G}_{k+1}(j+D^k\ell_1)|\cdots|\hat{G}_{k+1}(j+D^k\ell_d)),$$

then we obtain

$$\sum_{\ell\in\mathscr{R}_1}\hat{G}_{k+1}(j+D^k\ell)\, M_{k+1}(j+D^k\ell)\, \hat{H}_{k+1}(j+D^k\ell)^*=0; \tag{4.14}$$

that is, (4.9) holds.

It remains to show that the $\hat{G}_{k+1}(j+D^k\ell)$, $\ell\in\mathscr{R}_1$, defined above satisfy (4.7). Let

$$\begin{aligned}P:=(&\hat{H}_{k+1}(j+D^k\ell_1)\, M_{k+1}(j+D^k\ell_1)|\cdots|\\ &\hat{H}_{k+1}(j+D^k\ell_d)\, M_{k+1}(j+D^k\ell_d))=A^*,\end{aligned}$$

and let

$$S:=\operatorname{diag}(M_{k+1}(j+D^k\ell_1)^{-1}, ..., M_{k+1}(j+D^k\ell_d)^{-1}),$$

an $rd\times rd$ block diagonal matrix. Note that S is a positive definite Hermitian matrix, and by (4.14), we have $PQ_2^*=0$. Furthermore, since P and Q_2 have ranks r and $r(d-1)$ respectively, we see that the matrix $(\frac{P}{Q_2})$ is invertible. Hence, it follows from Lemma 4.1 that

$$\left(\frac{PS}{Q_2}\right)=\left(\begin{array}{c|c|c}\hat{H}_{k+1}(j+D^k\ell_1) & \cdots & \hat{H}_{k+1}(j+D^k\ell_d)\\ \hline \hat{G}_{k+1}(j+D^k\ell_1) & \cdots & \hat{G}_{k+1}(j+D^k\ell_d)\end{array}\right)$$

is invertible. Now, by (3.9), (4.6), and (4.14), we have

$$\left(\frac{PS}{Q_2}\right) \cdot \operatorname{diag}(M_{k+1}(j+D^k\ell_1), ..., M_{k+1}(j+D^k\ell_d)) \cdot \left(\frac{PS}{Q_2}\right)^*$$
$$= \left(\begin{array}{c|c} M_k(j) & 0 \\ \hline 0 & N_k(j) \end{array}\right).$$

Observe that the determinant of the product of matrices in the above identity is a positive real number. Consequently, we get

$$\det(M_k(j)) \det(N_k(j)) > 0,$$

and hence $\det(N_k(j)) > 0$. That is, (4.7) is satisfied. This completes the proof of the theorem. ▮

Remark 2. The proof of Theorem 4.1 gives an algorithm for obtaining $\hat{G}_{k+1} \in \mathscr{S}(D^{k+1})^{r(d-1)\times r}$, $k \geqslant 0$, which are clearly not unique. Thus, we have a constructive method for the construction of wavelet bases once we have a sequence of vectors $a_k \in \ell^2(\mathbb{Z}^s)^r$, $k \geqslant 0$, that gives a multiresolution $\{V_k : k \geqslant 0\}$ of $L^2([0, 2\pi)^s)$ via the sequence ϕ_k, $k \geqslant 0$, defined by $\hat{\phi}_k := a_k$.

In the case when the set

$$S_k = \{T_k^\ell \phi_k^m : m = 1, 2, ..., r, \ell \in \mathscr{L}_k\}$$

forms an orthonormal basis of V_k for each $k \geqslant 0$, we can actually construct an orthonormal wavelet basis for W_k. First of all, by Corollary 3.1, S_k being an orthonormal basis of V_k is equivalent to

$$M_k(j) = (\langle v_{k,j}^m, v_{k,j}^\mu \rangle)_{m,\mu=1}^r = \frac{1}{d^k} I_r$$

for each $j \in \mathscr{R}_k$, where I_r denotes the $r \times r$ identity matrix. Hence, (3.9) becomes

$$\sum_{\ell \in \mathscr{R}_1} \hat{H}_{k+1}(j+D^k\ell)\, \hat{H}_{k+1}(j+D^k\ell)^* = dI_r, \tag{4.15}$$

and (4.9) reduces to

$$\sum_{\ell \in \mathscr{R}_1} \hat{G}_{k+1}(j+D^k\ell)\, \hat{H}_{k+1}(j+D^k\ell)^* = 0. \tag{4.16}$$

Next, by an analogue of Corollary 3.1, we see that

$$\{T_k^\ell \psi_k^m : m = 1, 2, ..., r(d-1), \ell \in \mathscr{L}_k\},$$

where ψ_k^m is as defined in (4.3), forms an orthonormal basis for W_k if and only if

$$N_k(j) = (\langle u_{k,j}^m, u_{k,j}^\mu \rangle)_{m,\mu=1}^{r(d-1)} = \frac{1}{d^k} I_{r(d-1)}.$$

Thus, (4.6) yields

$$\sum_{\ell \in \mathscr{R}_1} \hat{G}_{k+1}(j + D^k \ell)\, \hat{G}_{k+1}(j + D^k \ell)^* = dI_{r(d-1)}. \tag{4.17}$$

Hence, the problem of constructing an orthonormal wavelet basis for W_k reduces to finding $\hat{G}_{k+1}$ satisfying the conditions (4.16) and (4.17). The following theorem gives an algorithm for obtaining such $\hat{G}_{k+1}(j)$.

THEOREM 4.2. *Suppose that for each $k \geqslant 0$ and $j \in \mathscr{R}_k$,* (4.15) *holds. Then there exist $\hat{G}_{k+1} \in \mathscr{S}(D^{k+1})^{r(d-1)\times r}$ such that the conditions* (4.16) *and* (4.17) *are satisfied for all $k \geqslant 0$ and $j \in \mathscr{R}_k$.*

Proof. Fix $k \geqslant 0$, $j \in \mathscr{R}_k$. Set

$$A := \frac{1}{\sqrt{d}} (\hat{H}_{k+1}(j + D^k \ell_1) | \cdots | \hat{H}_{k+1}(j + D^k \ell_d))^*.$$

Then (4.15) can be written as $A^*A = I_r$. Therefore the r columns of A form an orthonormal set in $\mathbb{C}^{rd}$. Hence, there exists a unitary $rd \times rd$ matrix Q such that

$$QA = \left(\frac{I_r}{0} \right).$$

Note that Q is an extension of A^*, so we can write Q as $Q = (\frac{A^*}{B})$ for some $r(d-1) \times rd$ matrix B. Now, we define $\hat{G}_{k+1}(j + D^k \ell)$, $\ell \in \mathscr{R}_1$, by

$$B = \frac{1}{\sqrt{d}} (\hat{G}_{k+1}(j + D^k \ell_1) | \cdots | \hat{G}_{k+1}(j + D^k \ell_d)).$$

Since Q is unitary, we have

$$QQ^* = \left(\begin{array}{c|c} A^*A & A^*B^* \\ \hline BA & BB^* \end{array} \right) = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & I_{r(d-1)} \end{array} \right).$$

It follows that

$$\sum_{\ell \in \mathscr{R}_1} \hat{G}_{k+1}(j+D^k\ell)\, \hat{G}_{k+1}(j+D^k\ell)^* = dI_{r(d-1)},$$

and

$$\sum_{\ell \in \mathscr{R}_1} \hat{G}_{k+1}(j+D^k\ell)\, \hat{H}_{k+1}(j+D^k\ell)^* = 0.$$

This finishes the proof of the theorem. ∎

Based on the algorithm in the proof of Theorem 4.1, we can construct matrices $\hat{G}_{k+1} \in \mathscr{S}(D^{k+1})^{r(d-1)\times r}$, $k \geqslant 0$, which lead to functions ψ_k^m, $k \geqslant 0$, $m = 1, 2, ..., r$, such that $\{T_k^\ell \psi_k^m : m = 1, 2, ..., r, \ell \in \mathscr{L}_k\}$ is a basis of W_k for every $k \geqslant 0$. A natural question one would ask is what are the choices of $\hat{G}_{k+1} \in \mathscr{S}(D^{k+1})^{r(d-1)\times r}$, $k \geqslant 0$, that yield a Riesz basis of the entire space $L^2([0, 2\pi)^s)$? The following theorem provides us with an answer.

THEOREM 4.3. *Suppose that there exist positive constants A and B such that*

$$\frac{A}{d^k} I_{r(d-1)} \leqslant \sum_{\ell \in \mathscr{R}_1} \hat{G}_{k+1}(j+D^k\ell)\, M_{k+1}(j+D^k\ell)\, \hat{G}_{k+1}(j+D^k\ell)^* \leqslant \frac{B}{d^k} I_{r(d-1)}, \tag{4.18}$$

for all $k \geqslant 0$ and $j \in \mathscr{R}_k$. Then the collection of functions

$$\{\phi_0^m : m = 1, 2, ..., r\} \cup \{T_k^\ell \psi_k^m : k \geqslant 0, m = 1, 2, ..., r(d-1), \ell \in \mathscr{L}_k\} \tag{4.19}$$

forms a Riesz basis of $L^2([0, 2\pi)^s)$.

Remark 3. It follows from (4.6) that (4.18) is equivalent to

$$\frac{A}{d^k} \|x\|^2 \leqslant x N_k(j)\, x^* \leqslant \frac{B}{d^k} \|x\|^2, \tag{4.20}$$

for all $k \geqslant 0$, $j \in \mathscr{R}_k$ and $x = (x_1, ..., x_{r(d-1)}) \in \mathbb{C}^{r(d-1)}$.

Proof of Theorem 4.3. For $k \geqslant 0$, $j \in \mathscr{R}_k$, since $N_k(j) := (\langle u_{k,j}^m, u_{k,j}^\mu \rangle)_{m,\mu=1}^{r(d-1)}$ is positive definite, the linearly independent set $\{u_{k,j}^m : m = 1, 2, ..., r(d-1)\}$ can be orthonormalized to give an orthonormal set $\{\tilde{u}_{k,j}^m : m = 1, 2, ..., r(d-1)\}$ by

$$\tilde{u}_{k,j} := N_k(j)^{-1/2}\, u_{k,j},$$

where $u_{k,j}=(u^1_{k,j}, ..., u^{r(d-1)}_{k,j})^T$ and $\tilde{u}_{k,j}=(\tilde{u}^1_{k,j}, ..., \tilde{u}^{r(d-1)}_{k,j})^T$ (see [21, pp. 25–26]). Thus $\{\tilde{u}^m_{k,j}: m=1, 2, ..., r(d-1), j\in\mathscr{R}_k\}$ forms an orthonormal basis of W_k for every $k\geqslant 0$. Similarly, $\{\tilde{v}^m_{0,0}: m=1, 2, ..., r\}$, where $\tilde{v}_{0,0}:=M_0(0)^{-1/2}\, v_{0,0}$, forms an orthonormal basis of V_0. Then it follows from the orthogonal decomposition

$$L^2([0, 2\pi)^s)=V_0\oplus^{\perp}W_0\oplus^{\perp}W_1\oplus^{\perp}\cdots$$

that the collection

$$\{\tilde{v}^m_{0,0}: m=1, 2, ..., r\}\cup\{\tilde{u}^m_{k,j}: k\geqslant 0, m=1, 2, ..., r(d-1), j\in\mathscr{R}_k\}$$

is an orthonormal basis of $L^2([0, 2\pi)^s)$.

Since $|\mathscr{L}_k|=|\mathscr{R}_k|$ for $k\geqslant 0$, there exists a bijection $\tau_k: \mathscr{L}_k\to\mathscr{R}_k$ for every $k\geqslant 0$. Now, define a linear operator S on $L^2([0, 2\pi)^s)$ by

$$S(f):=\hat{p}_0(0)\, M_0(0)^{1/2}\, \tilde{v}_{0,0}+\sum_{k=0}^{\infty}\sum_{j\in\mathscr{R}_k}\hat{q}_k(j)\, N_k(j)^{1/2}\, \tilde{u}_{k,j},$$
$$f\in L^2([0, 2\pi)^s), \tag{4.21}$$

where $\hat{p}_0(0):=(\langle f, \tilde{v}^1_{0,0}\rangle, ..., \langle f, \tilde{v}^r_{0,0}\rangle)$, and

$$\hat{q}_k(j):=\left(\sum_{\ell\in\mathscr{L}_k}\langle f, \tilde{u}^1_{k,\tau_k(\ell)}\rangle\, e^{-ij\cdot(2\pi M^{-k}\ell)}, ..., \sum_{\ell\in\mathscr{L}_k}\langle f, \tilde{u}^{r(d-1)}_{k,\tau_k(\ell)}\rangle\, e^{-ij\cdot(2\pi M^{-k}\ell)}\right).$$

The operator S is well defined. Indeed, for $f\in L^2([0, 2\pi)^s)$, we have

$$\begin{aligned}&\|\hat{p}_0(0)\|^2+\sum_{k=0}^{\infty}\sum_{j\in\mathscr{R}_k}\frac{1}{d^k}\|\hat{q}_k(j)\|^2\\&\quad=\sum_{m=1}^{r}|\langle f, \tilde{v}^m_{0,0}\rangle|^2+\sum_{k=0}^{\infty}\sum_{m=1}^{r(d-1)}\sum_{j\in\mathscr{R}_k}|\langle f, \tilde{u}^m_{k,j}\rangle|^2=\|f\|^2<\infty,\end{aligned} \tag{4.22}$$

where $\hat{p}_0(0)=(\hat{p}^1_0(0), ..., \hat{p}^r_0(0))$, $\hat{q}_k(j)=(\hat{q}^1_k(j), ..., \hat{q}^{r(d-1)}_k(j))$, and the relation (1.7) is used to establish

$$\begin{aligned}\frac{1}{d^k}\sum_{j\in\mathscr{R}_k}|\hat{q}^m_k(j)|^2&=\sum_{\ell\in\mathscr{L}_k}|\langle f, \tilde{u}^m_{k,\tau_k(\ell)}\rangle|^2\\&=\sum_{j\in\mathscr{R}_k}|\langle f, \tilde{u}^m_{k,j}\rangle|^2, \qquad m=1, 2, ..., r(d-1).\end{aligned}$$

Since $M_0(0)$ is a positive definite constant matrix, all its eigenvalues $\lambda_1, ..., \lambda_r$ are real and positive and,

$$A_0 I_r\leqslant M_0(0)\leqslant B_0 I_r, \tag{4.23}$$

where $A_0 := \min\{\lambda_1, ..., \lambda_r\}$ and $B_0 := \max\{\lambda_1, ..., \lambda_r\}$. Thus

$$\begin{aligned} \|S(f)\|^2 &= \hat{p}_0(0)\, M_0(0)\, \hat{p}_0(0)^* + \sum_{k=0}^{\infty} \sum_{j \in \mathcal{R}_k} \hat{q}_k(j)\, N_k(j)\, \hat{q}_k(j)^* \\ &\leqslant B_0\, \|\hat{p}_0(0)\|^2 + \sum_{k=0}^{\infty} \sum_{j \in \mathcal{R}_k} \frac{B}{d^k} \|\hat{q}_k(j)\|^2, \end{aligned} \tag{4.24}$$

which is finite by (4.22).

Next, for $k \geqslant 0$, $m = 1, 2, ..., r(d-1)$ and $j \in \mathcal{R}_k$, it follows from (4.3) that

$$S(\tilde{u}^m_{k,j}) = \sum_{v \in \mathcal{R}_k} u^m_{k,v} e^{-iv \cdot (2\pi M^{-k} \tau_k^{-1}(j))} = T_k^{\tau_k^{-1}(j)} \psi^m_k.$$

Also, for $m = 1, 2, ..., r$, $S(\tilde{v}^m_{0,0}) = \phi^m_0$.

It remains to show that S is an isomorphism. Let

$$f := \sum_{m=1}^{r} \langle f, \tilde{v}^m_{0,0} \rangle\, \tilde{v}^m_{0,0} + \sum_{k=0}^{\infty} \sum_{m=1}^{r(d-1)} \sum_{j \in \mathcal{R}_k} \langle f, \tilde{u}^m_{k,j} \rangle\, \tilde{u}^m_{k,j}$$

be any function in $L^2([0, 2\pi)^s)$. Then it follows from (4.20), (4.22)–(4.24) that

$$\min\{A_0, A\}\, \|f\|^2 \leqslant \|S(f)\|^2 \leqslant \max\{B_0, B\}\, \|f\|^2.$$

Thus S is an isomorphism, and the collection

$$\{\phi^m_0 : m = 1, 2, ..., r\} \cup \{T_k^{\tau_k^{-1}(j)} \psi^m_k : k \geqslant 0, m = 1, 2, ..., r(d-1), j \in \mathcal{R}_k\}$$

forms a Riesz basis of $L^2([0, 2\pi)^s)$. Since $\tau_k : \mathcal{L}_k \to \mathcal{R}_k$ is a bijection for all $k \geqslant 0$, we conclude that the collection (4.19) is a Riesz basis of $L^2([0, 2\pi)^s)$. ∎

It should be mentioned that in the orthonormal case handled by Theorem 4.2, the condition (4.18) is automatically satisfied with $A = B = 1$, and the corresponding collection of functions (4.19) forms an orthonormal basis of $L^2([0, 2\pi)^s)$.

5. AN EXAMPLE

Let $M = D = 2I_2$. Then $\mathcal{L}_k = \mathcal{R}_k = \{(\ell_1, \ell_2) \in \mathbb{Z}^2 : \ell_1, \ell_2 = 0, 1..., 2^k - 1\}$. For $k = 0, 1, ...,$ and $j = (j_1, j_2) \in \mathcal{R}_{k+1}$, define $\hat{H}_{k+1} \in \mathcal{S}(D^{k+1})$ by

$$\hat{H}_{k+1}(j) = (\cos 2^{-k-1} \pi j_1)^{m_1} (\cos 2^{-k-1} \pi j_2)^{m_2} (\cos 2^{-k-1} \pi (j_1 + j_2))^{m_3}, \tag{5.1}$$

where m_1, m_2, m_3 are positive integers. Then for $n \in \mathbb{Z}^2$,

$$\sum_{\ell=1}^{\infty} |\hat{H}_\ell(n) - 1| < \infty, \tag{5.2}$$

and for $k \geqslant 0$,

$$\sum_{\ell \in \mathcal{R}_1} |\hat{H}_{k+1}(j + 2^k \ell)|^2 \leqslant 1, \qquad j \in \mathcal{R}_k. \tag{5.3}$$

Furthermore, it follows from the relation

$$\prod_{\ell=1}^{\infty} (\cos 2^{-\ell} z)^m = \left(\frac{\sin z}{z}\right)^m, \qquad z \in \mathbb{C},$$

that for $n = (n_1, n_2) \in \mathbb{Z}^2$,

$$\prod_{\ell=k+1}^{\infty} \hat{H}_\ell(n) = \left(\frac{\sin 2^{-k}\pi n_1}{2^{-k}\pi n_1}\right)^{m_1} \left(\frac{\sin 2^{-k}\pi n_2}{2^{-k}\pi n_2}\right)^{m_2} \left(\frac{\sin 2^{-k}\pi(n_1+n_2)}{2^{-k}\pi(n_1+n_2)}\right)^{m_3}. \tag{5.4}$$

Thus

$$\hat{\phi}_k(n) := \prod_{\ell=k+1}^{\infty} \hat{H}_\ell(n), \qquad n \in \mathbb{Z}^2, \tag{5.5}$$

is the Fourier sequence of a function $\phi_k \in L^2([0, 2\pi)^2)$.

Now for $k \geqslant 0$, $j \in \mathcal{R}_k$,

$$M_k(j) = \sum_{p \in \mathbb{Z}^2} |\hat{\phi}_k(j + 2^k p)|^2. \tag{5.6}$$

To show that $M_k(j) > 0$ for all $j \in \mathcal{R}_k$, first note that for $j = (j_1, j_2) \in \mathcal{R}_k$, if $j_1 + j_2 < 2^k$, then

$$|\hat{\phi}_k(j)|^2 = \left(\frac{\sin 2^{-k}\pi j_1}{2^{-k}\pi j_1}\right)^{2m_1} \left(\frac{\sin 2^{-k}\pi j_2}{2^{-k}\pi j_2}\right)^{2m_2} \left(\frac{\sin 2^{-k}\pi(j_1+j_2)}{2^{-k}\pi(j_1+j_2)}\right)^{2m_3} > 0.$$

On the other hand, if $j_1 + j_2 \geqslant 2^k$, then $j_1 \neq 0$, $j_2 \neq 0$, $0 \leqslant j_1 + j_2 - 2^k < 2^k$ and $-2^k < j_1 - 2^k < 0$. Thus with $p = (-1, 0)$,

$$\begin{aligned} |\hat{\phi}_k(j + 2^k p)|^2 &= \left(\frac{\sin 2^{-k}\pi(j_1 - 2^k)}{2^{-k}\pi(j_1 - 2^k)}\right)^{2m_1} \left(\frac{\sin 2^{-k}\pi j_2}{2^{-k}\pi j_2}\right)^{2m_2} \\ &\quad \times \left(\frac{\sin 2^{-k}\pi(j_1 + j_2 - 2^k)}{2^{-k}\pi(j_1 + j_2 - 2^k)}\right)^{2m_3} > 0. \end{aligned}$$

Therefore for all $j \in \mathscr{R}_k$, $M_k(j) > 0$. This shows that $\{T_k^{\ell}\phi_k : \ell \in \mathscr{L}_k\}$ is linearly independent.

By (5.4) and (5.5), for any $n \in \mathbb{Z}^2$,

$$\hat{\phi}_k(n) \to 1 \qquad \text{as} \quad k \to \infty.$$

Therefore,

$$\{n \in \mathbb{Z}^2 : \hat{\phi}_k(n) = 0 \text{ for all } k \geqslant 0\} = \emptyset.$$

Hence the functions ϕ_k, $k \geqslant 0$, generate a periodic multiresolution $\{V_k : k \geqslant 0\}$ of $L^2([0, 2\pi)^2)$.

To construct wavelet bases for the orthogonal complement W_k of V_k in V_{k+1}, it suffices to find matrices $\hat{G}_{k+1} \in \mathscr{S}(D^{k+1})^{3\times 1}$ satisfying (4.7) and (4.9) for all $j \in \mathscr{R}_k$. They are readily obtained using the algorithm described in the proof of Theorem 4.1.

In order to describe the matrices $\hat{G}_{k+1}$, we define the set

$$B := \{(0, 0, 0)^T, (1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\},$$

and consider a bijection $e: \mathscr{R}_1 \to B$ with $e(0) = 0$. For $j \in \mathscr{R}_k$, let $\alpha_{k,j}(q)$, $q \in \mathscr{R}_1$, be arbitrary nonzero complex numbers. Since $M_k(j) > 0$, we can choose an $\ell \in \mathscr{R}_1$ such that $\hat{H}_{k+1}(j + 2^k\ell) \neq 0$. Then for $q \in \mathscr{R}_1$,

$$\hat{G}_{k+1}(j + 2^k q) = \begin{cases} -\displaystyle\sum_{p \in \mathscr{R}_1} \frac{\alpha_{k,j}(p-\ell)\, \hat{H}_{k+1}(j+2^k p)\, M_{k+1}(j+2^k p)}{\hat{H}_{k+1}(j+2^k\ell)\, M_{k+1}(j+2^k\ell)}\, e(p-\ell), & q = \ell, \\ \alpha_{k,j}(q-\ell)\, e(q-\ell), & q \neq \ell. \end{cases}$$

ACKNOWLEDGMENT

We thank an anonymous referee for suggesting the results in Theorem 2.2 and Corollary 2.3.

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